Exercises (recommended)

1. Recall that we stated a hypergraph version of the Kővári–Sós–Turán theorem, and proved it (at least in the case $k = 3$) by induction on k. Try proving the $k = 2$ case (i.e. the original Kővári–Sós–Turán theorem) via a similar inductive approach. What does the $k = 1$ case even mean?

2. In class, we only proved the hypergraph Kővári–Sós–Turán theorem in only one special case, namely for $k = 3$ and $s_1 = s_2 = s_3$. Prove the general result, namely that

$$\mathrm{ex}(n, K^{(k)}_{s_1,\dots,s_k}) \leqslant O\left(n^{k-\frac{1}{s_1 s_2 \cdots s_{k-1}}}\right).$$

3. For a k-graph $\mathcal{H}$ and an integer n, let $\pi_n(\mathcal{H}) \coloneqq \mathrm{ex}(n, \mathcal{H})/\binom{n}{k}$.

   ⋆(a) Prove that $\pi_n(\mathcal{H}) \geqslant \pi_{n+1}(\mathcal{H})$ for all n. Conclude that $\pi(\mathcal{H}) \coloneqq \lim_{n\to\infty} \pi_n(\mathcal{H})$ is well-defined. $\pi(\mathcal{H})$ is called the *Turán density* of $\mathcal{H}$.

   (b) Let H be a graph (i.e. $k = 2$). Find a formula for $\pi(H)$.

4. In this problem, you will prove a supersaturation result for complete bipartite graphs.

   (a) Given two graphs H, G, a *graph homomorphism* from H to G is a function $f : V(H) \to V(G)$ with the property that if uv is an edge of H, then $f(u)f(v)$ is an edge of G. Note that if f is injective, then this yields a copy of H in G. If f is not injective, we say this is a *pseudocopy*.

   Prove that if $v(G) = n$, then there are at most $n^{v(H)}$ homomorphisms from H to G, and at most $\binom{v(H)}{2} n^{v(H)-1}$ pseudocopies of H in G.

   (b) Suppose G has n vertices and $pn^2/2$ edges (we say that G has *edge density* p). Prove[1] that there are at least $p^t n^{1+t}$ homomorphisms from $K_{1,t}$ to G.

   ⋆(c) Suppose G has n vertices and $pn^2/2$ edges (we say that G has *edge density* p). Prove[2] that there are at least $p^{st} n^{s+t}$ homomorphisms from $K_{s,t}$ to G.

   (d) Deduce from parts (a) and (b) the following supersaturation result. For every $\varepsilon > 0$ and integers s, t, there exists a $\delta > 0$ so that the following holds for sufficiently large n. If G has n vertices and $\varepsilon\binom{n}{2}$ edges, then G has at least $\delta\binom{n}{s+t}$ copies of $K_{s,t}$.

   ⋆(e) Can you prove analogous results for k-uniform hypergraphs?

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.
[1] *Hint:* Jensen's inequality.
[2] *Hint:* Use Jensen's inequality again!

Problems (optional)

$\star$ 1. Recall that $K_r^{(k)}$ denotes the complete k-uniform hypergraph with r vertices.

(a) Prove[3] that $\mathrm{ex}(n, K_4^{(3)}) \geqslant (\frac{5}{9} + o(1))\binom{n}{3}$.

$\star$ (b) Prove that $\mathrm{ex}(n, K_r^{(3)}) \geqslant (1 - (\frac{2}{r-1})^2 + o(1))\binom{n}{3}$ for all $r \geqslant 4$.

(c) Prove that

$$\mathrm{ex}(n, K_r^{(k)}) \leqslant \left(1 - \frac{1}{\binom{r}{k}} + o(1)\right)\binom{n}{k}.$$

$\star\star$ (d) Prove the best known upper bound on $\mathrm{ex}(n, K_r^{(k)})$, namely

$$\mathrm{ex}(n, K_r^{(k)}) \leqslant \left(1 - \frac{1}{\binom{r-1}{k-1}} + o(1)\right)\binom{n}{k}.$$

? (e) Improve any of the bounds above.

2. In this problem, you will study the extremal number of the graph of the 3-dimensional cube, denoted Q_3.

(a) Using Exercise 4, prove that if G is an n-vertex graph with $e(G) \geqslant \Omega(n^{3/2})$, then G has at least $\Omega(p^4 n^4)$ copies of C_4, where $p = 2e(G)/n^2$ is the edge density of G.

(b) Define the *C_4-graph* $C_4(G)$ of G to be the following graph. Its vertices are the *edges* of G, and two such are adjacent in $C_4(G)$ if they are the opposite sides of a C_4 in G. Relate copies of C_4 in $C_4(G)$ to copies of Q_3 in G.
Be careful! Not every C_4 in $C_4(G)$ corresponds to a Q_3 in G; figure out why not.

(c) Prove that $\mathrm{ex}(n, Q_3) \leqslant O(n^{8/5})$.

? (d) Prove a matching lower bound, $\mathrm{ex}(n, Q_3) \geqslant \Omega(n^{8/5})$.

3. In this problem, you will prove a slightly weaker version of Turán's theorem using a technique called *Lagrangians* (or the *Motzkin–Straus inequality*).

(a) Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. Define the *graph polynomial*

$$p_G(x_1, \ldots, x_n) = \sum_{v_i v_j \in E(G)} x_i x_j$$

and define the *Lagrangian* $\lambda(G)$ of G to be the maximum of $p_G(x_1, \ldots, x_n)$ over all vectors $(x_1, \ldots, x_n)$ satisfying $x_i \geqslant 0$ for all i, and $\sum_{i=1}^n x_i = 1$. Prove that this maximum is well-defined.

[3] *Hint:* Split the vertex set into three equal-sized parts.

(b) Prove that $\lambda(G) \geqslant e(G)/n^2$.

$\star$ (c) Let $x = (x_1, \dots, x_n)$ be a point achieving the maximum in $\lambda(G)$, i.e. a vector with $p_G(x_1, \dots, x_n) = \lambda(G)$. Moreover, assume that the number of non-zero entries in x is minimized among all such maximizers. Prove that the set of non-zero coordinates in x forms a clique in G.

(d) Deduce from the previous part that if G is K_r-free, then $\lambda(G) \leqslant \frac{1}{2}(1 - \frac{1}{r-1})$.

(e) Conclude that if G is an n-vertex K_r-free graph, then $e(G) \leqslant (1 - \frac{1}{r-1})\frac{n^2}{2}$. Note that this is slightly weaker than the bound in Turán's theorem, but matches it if $r - 1$ divides n.

$\star\star$ 4. In class we proved the following sampling lemma: If G is an n-vertex graph with $e(G) \geqslant \beta\binom{n}{2}$, then the number of m-sets of vertices M with $e(M) \geqslant \alpha\binom{m}{2}$ is at least $(\beta - \alpha)\binom{n}{m}$. In fact, the proof showed that we could replace $\beta - \alpha$ above with $(\beta - \alpha)/(1 - \alpha)$.

Is this result best possible, or close to best possible? That is, is it true that for arbitrarily large n, there exists some n-vertex graph with $e(G) \approx \beta\binom{n}{2}$ and roughly $\frac{\beta-\alpha}{1-\alpha}\binom{n}{m}$ m-sets M with $e(M) \geqslant \alpha\binom{n}{m}$?

For concreteness, feel free to fix your favorite values of α, β, e.g. $\alpha = 1/3$ and $\beta = 2/3$. So can you find a sequence of graphs with around $\frac{2}{3}\binom{n}{2}$ edges so that roughly $\frac{1}{2}\binom{n}{m}$ of the m-sets M satisfy $e(M) \geqslant \frac{1}{3}\binom{m}{2}$?