Exercises (recommended)

⋆ 1. Recall that the *Turán density* of a k-graph $\mathcal{H}$ is $\pi(\mathcal{H}) := \lim_{n\to\infty} \operatorname{ex}(n,\mathcal{H})/\binom{n}{k}$. Prove the following general form of the supersaturation theorem.

For every k-graph $\mathcal{H}$ and every $\varepsilon > 0$, there exists some $\delta > 0$ so that the following holds for all sufficiently large n. If $\mathcal{G}$ is an n-vertex k-graph with

$$e(\mathcal{G}) \geqslant (\pi(\mathcal{H}) + \varepsilon)\binom{n}{k}$$

then $\mathcal{G}$ has at least $\delta\binom{n}{v(\mathcal{H})}$ copies of $\mathcal{H}$.

2. For a graph H and an integer s, we denote by $H[s]$ the *s-blowup* of H. This is the graph obtained by replacing every vertex of H by an independent set of size s, and replacing every edge of s by a copy of $K_{s,s}$. Similarly, if $\mathcal{H}$ is a k-graph, then $\mathcal{H}[s]$ is the k-graph obtained by replacing every vertex by s vertices, and replacing every edge by a copy of $K^{(k)}_{s,\ldots,s}$.

   (a) Check that if $H = K_k$, our two definitions of $K_k[s]$ coincide.

   (b) Deduce from the previous problem the following general form of the Erdős–Stone theorem.

   For every k-graph $\mathcal{H}$ and every positive integer s, we have $\pi(\mathcal{H}[s]) = \pi(\mathcal{H})$.

3. (a) Let $\mathcal{H}, \mathcal{G}$ be k-graphs. A *homomorphism* $\mathcal{H} \to \mathcal{G}$ is a function $V(\mathcal{H}) \to V(\mathcal{G})$ which maps hyperedges to hyperedges. We say that $\mathcal{G}$ is *$\mathcal{H}$-hom-free* if there is no homomorphism $\mathcal{H} \to \mathcal{G}$.

   Let $\operatorname{ex}_{\text{hom}}(n,\mathcal{H})$ denote the maximum number of hyperedges in an $\mathcal{H}$-hom-free n-vertex k-graph. Prove that $\operatorname{ex}_{\text{hom}}(n,\mathcal{H}) \leqslant \operatorname{ex}(n,\mathcal{H})$.

   (b) Now let H be a graph (i.e. let $k = 2$). Determine $\operatorname{ex}_{\text{hom}}(n, H)$.

   (c) Prove that the limit $\pi_{\text{hom}}(\mathcal{H}) := \lim_{n\to\infty} \operatorname{ex}_{\text{hom}}(n,\mathcal{H})/\binom{n}{k}$ exists.

   (d) Prove that $\pi_{\text{hom}}(\mathcal{H}) = \pi(\mathcal{H})$.

4. Prove that if $\mathcal{H}$ is a k-graph, then either $\pi(\mathcal{H}) = 0$ or $\pi(\mathcal{H}) \geqslant k!/k^k$.

Problems (optional)

1. Let G be a graph, and recall that $\alpha(G)$ denotes its independence number.

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

(a) By picking a random permutation of $V(G)$, prove that

$$\alpha(G) \geqslant \sum_{v \in V(G)} \frac{1}{\deg(v)+1}.$$

(b) Apply Jensen's inequality to conclude that

$$\alpha(G) \geqslant \frac{n}{d+1},$$

where d is the average degree of G. Recall that you already proved this result, as a consequence of Turán's theorem.

⋆ (c) By more carefully analyzing the proof above, give an alternative proof of Turán's theorem.

⋆ 2. (a) By more carefully analyzing the proof we saw in class, prove the following strengthening of the Erdős–Stone–Simonovits theorem. For every graph H, there exists some $\delta > 0$ such that

$$\mathrm{ex}(n, H) \leqslant t_{\chi(H)-1}(n) + O(n^{2-\delta}).$$

(b) Prove the following converse: for every $\delta > 0$ and every $r \geqslant 2$, there exists a graph H with $\chi(H) = r$ and

$$\mathrm{ex}(n, H) \geqslant t_{r-1}(n) + \Omega(n^{2-\delta}).$$

⋆⋆ 3. In this problem, you will prove the following amazing strengthening of the Kővári–Sós–Turán theorem: if H is a bipartite graph and every vertex on one side has degree at most s, then $\mathrm{ex}(n, H) = O(n^{2-1/s})$.

⋆⋆ (a) Prove[1] the following lemma. For all positive integers a, b, there exists some constant $C > 0$ such that the following holds. Let G be an n-vertex graph with average degree $d \geqslant Cn^{1-1/s}$. Then there exists $U \subseteq V(G)$ with $|U| \geqslant a$ so that every s-tuple of vertices in U has at least $a + b$ common neighbors.

(b) Using the lemma, prove that $\mathrm{ex}(n, H) \leqslant O(n^{2-1/s})$ if every vertex on one side of H has degree at most s.

[1] *Hint:* Pick $x_1, \ldots, x_s$ to be uniformly random vertices of G, chosen with repetition, and let X be the common neighborhood of $x_1, \ldots, x_s$. The desired U can be obtained by deleting some vertices from X, with positive probability.