Exercises (recommended)

1. (a) Prove that $r(3,3) = 6$.
   (b) Prove that $r(3,4) = 9$.
   (c) Prove that $r(4,4) \leqslant 18$.
   $\star$ (d) Prove that $r(4,4) = 18$.
   ? (e) The best known bounds on $r(5,5)$ are $43 \leqslant r(5,5) \leqslant 46$. Can you improve either of these bounds?
   $\star$ (f) Prove that $r(3;3) = 17$ (recall that $r(3;3)$ is the 3-color Ramsey number).

2. (a) Prove that
   $$r(3;q) \leqslant 1 + q!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right).$$
   (b) Conclude that $r(3;q) \leqslant \lceil e \cdot q! \rceil$, where e is Euler's constant.

3. (a) Using the fact that $r(k) < 4^k$, prove that $r(k;q) < 4^{4^{4^{\cdot^{\cdot^{\cdot^{k}}}}}}$, where the number of 4s is $\lceil \log_2 q \rceil$.
   (b) Prove Theorem 13.5 in the notes. In particular, derive the bound $r(k;q) < q^{qk}$, which is much stronger than that in part (a).

4. Prove the following supersaturation version of Ramsey's theorem, which is usually called a *Ramsey multiplicity* result.

   For all positive integers k, q, there exists some $\delta > 0$ so that the following holds for every sufficiently large N. No matter how we q-color the edges of K_N, there are at least $\delta\binom{N}{k}$ monochromatic copies of K_k.

Problems (optional)

✣ 1. (a) Prove that for any positive integer q, there exists a positive integer $N = N(q)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x, y, z \in [\![N]\!]$ such that $x, y, z, x+y, y+z, x+y+z$ all receive the same color. (Note that $x+z$ is omitted!)
   (b) Generalize the previous part as follows. Prove that for all positive integers q, t, there exists a positive integer $N = N(q,t)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x_1, \ldots, x_t \in [\![N]\!]$ such that the sums $\sum_{i=a}^{b} x_i$ all receive the same color, for all non-empty $1 \leqslant a \leqslant b \leqslant t$.

$\star$ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

⋆ (c) Prove that in part (b), one can moreover ensure that the numbers $x_1, \dots, x_t$ are all distinct.

2. Let H be a graph. The 2-*colored extremal number* of H, denoted $\mathrm{ex}_2(n, H)$, is defined to be the maximum number of edges in an n-vertex graph G for which there exists a 2-coloring of $E(G)$ containing no monochromatic copy of H.

   Find an exact formula for $\mathrm{ex}_2(n, K_k)$.

⋆ 3. Let $f, g_1, \dots, g_q : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that there exist $\varepsilon, \delta > 0$ such that whenever $x, y \in \mathbb{R}$ satisfy $f(x) - f(y) \geqslant \varepsilon$, then

$$\max_{i \in [\![q]\!]} (g_i(x) - g_i(y)) \geqslant \delta.$$

   Prove that if $g_1, \dots, g_q$ are all bounded, then f is bounded as well.

✣⋆ 4. In class, we proved that $r(k) < 4^k$ using the Erdős–Szekeres argument. Ramsey's original proof used a *different* argument, which yielded the worse bound $r(k) \leqslant k!$. Find a natural argument yielding this bound. (That is, don't simply quote or rederive the Erdős–Szekeres argument!)