Exercises (recommended)

1. Prove that the 3-uniform Ramsey number satisfies

$$r_3(k) \geqslant 2^{ck^2}$$

for some absolute constant $c > 0$.

2. In a red/blue coloring of $E(K_N)$, denote by $\deg_R(v), \deg_B(v)$ the red and blue degrees, respectively, of a vertex v.

   (a) Prove that the number of monochromatic triangles in such a coloring is equal to

$$\frac{1}{2}\left(\sum_{v \in V(K_N)} \left[\binom{\deg_R(v)}{2} + \binom{\deg_B(v)}{2}\right] - \binom{N}{3}\right).$$

   (b) Prove that every 2-coloring of $E(K_6)$ contains at least two monochromatic triangles, and in particular obtain a new proof that $r(3) \leqslant 6$.

   ⋆(c) Prove that every 2-coloring of $E(K_N)$ contains at least $\frac{N(N-1)(N-5)}{24}$ monochromatic triangles.

3. On yesterday's homework, you proved a supersaturation version of Ramsey's theorem. In the 2-color case, it states that for every $k \geqslant 3$, there is some $\delta > 0$ such that for every sufficiently large N, every 2-coloring of $E(K_N)$ contains at least $\delta\binom{N}{k}$ monochromatic copies of K_k.

   The *Ramsey multiplicity constant* of K_k, denoted $c(K_k)$, is defined to be the supremum of all δ for which this statement is true.

   (a) Prove that $c(K_k) \geqslant \binom{r(k)}{k}^{-1}$. Conclude that $c(K_k) \geqslant 4^{-k^2}$.

   (b) Prove that $c(K_k) \leqslant 2^{1-\binom{k}{2}}$.

   (c) Prove[1] that $c(K_3) = \frac{1}{4}$.

4. For every $N \geqslant 3$, give N points in $\mathbb{R}^2$ with no three in convex position. This shows that the assumption in Theorem 14.5 that no three points are collinear is necessary.

5. Prove that for every $k \geqslant 3$, there exists some N so that the following holds. Among any N points in the plane, there are either k points lying on a line, or k points in convex position.

[1] *Hint:* Use the previous exercise.

6. Let $N = r_3(k)$, and let $p_1, \ldots, p_N$ be points in $\mathbb{R}^2$ with no three collinear. Define $\chi : E(K_N^{(3)}) \to \{\text{even}, \text{odd}\}$ by

$$\chi(\{i,j,\ell\}) := \begin{cases} \text{even} & \text{if there are an even number of points } p_m \text{ in the triangle } p_i p_j p_\ell, \\ \text{odd} & \text{otherwise.} \end{cases}$$

Prove that a monochromatic $K_k^{(3)}$ under χ corresponds to k points in convex position. Conclude that $\mathrm{Kl}(k) \leqslant r_3(k)$, and in particular obtain a new proof of Theorem 14.5.

Problems (optional)

1. (a) By more carefully analyzing the proof of Theorem 13.7 in the notes, prove that

$$r(k) > \left(\frac{1}{e\sqrt{2}} - o(1)\right) k 2^{k/2}.$$

$\star$ (b) Improve this bound by a constant factor.

? (c) Improve this bound by a super-constant factor.

$\star$ 2. Prove that $\mathrm{Kl}(5) = 9$.

3. Given two graphs G, H, their *lexicographic product* $G \cdot H$ is defined as follows. Its vertex set is $V(G \cdot H) = V(G) \times V(H)$, and two vertices $(a,b), (c,d)$ are adjacent if either $ac \in E(G)$ or $a = c$ and $bd \in E(H)$.

(a) Compute the size of the largest clique and the largest independent set in $G \cdot H$.

(b) Prove that the Ramsey number $r(k)$ satisfies $r(k+1) > k^{\log_2(5)}$ for all k that are powers of 2.

[Note that this already disproves Turán's belief that $r(k)$ may grow only quadratically as a function of k.]

$\star\star$ (c) Using the same approach, find an *explicit* construction of a coloring witnessing that $r(k)$ grows super-polynomially in k. In other words, for any $C > 0$ and any sufficiently large k, find an explicit 2-coloring of $E(K_N)$, where $N = k^C$, with no monochromatic clique of order k.

? (d) Can you use such an approach to resolve Open problem 13.8?

4. A collection of points in $\mathbb{R}^d$ is said to be *in general position* if no $d+1$ of them lie on a $(d-1)$-dimensional hyperplane. (So in two dimensions, this says that no three are collinear, in three dimensions it says that no four are coplanar, etc.)

(a) Prove that among any $d+3$ points in $\mathbb{R}^d$ which are in general position, there are $d+2$ in convex position.

(b) Given $k \geqslant d+2$, let $N = r_{d+2}(d+3, k)$. Prove that among any N points in $\mathbb{R}^d$ in general position, there are k in convex position.

(c) Prove that among any $\mathrm{Kl}(k)$ points in $\mathbb{R}^d$, no three collinear, there are k in convex position.

[This is stronger than the result in (b) in two ways: the bound is *independent* of d, and the assumption is weakened from general position to no three collinear.]

✣ 5. A *subdivision* of a graph H is obtained from H by replacing every edge of H by a path of some length (not necessarily the same length for all edges, and paths of length 1 are allowed, so that H is a subdivision of itself). A famous conjecture of Hajós asserts that if $\chi(G) \geqslant k$, then G contains a subdivision of K_k as a subgraph.

(a) Prove that Hajós' conjecture is true for $k \leqslant 3$.

⋆ (b) Prove that Hajós' conjecture is true for $k = 4$.

(c) Prove that Hajós' conjecture for $k = 5$ implies the four-color theorem. Conclude that it is probably pretty hard to prove the $k = 5$ case.

(d) Prove that if Hajós' conjecture is true, then $r(k) \leqslant 3k^3$. Conclude that Hajós' conjecture is false.

✣ 6. A classical fact in graph theory is that there exist triangle-free graphs of arbitrarily high chromatic number. In this exercise, you will see a non-standard Ramsey-theoretic proof.

For an integer N, let S_N be a graph with vertex set $\binom{[\![N]\!]}{2}$, where we think of the vertices of S_N as ordered pairs (a, b) with $1 \leqslant a < b \leqslant N$. The edges of S_N consist of all pairs of the form $((a, b), (b, c))$ for $a < b < c$.

(a) Prove that S_N is triangle-free.

(b) Prove that $\chi(S_N) \to \infty$ as $N \to \infty$.

✣ 7. (a) Let $K_{\mathbb{N}}$ denote the complete graph whose vertex set is $\mathbb{N}$. Prove the "infinite Ramsey theorem": for any positive integer q, and any q-coloring of $K_{\mathbb{N}}$, there is an infinite monochromatic clique.

(b) State and prove the infinite *hypergraph* Ramsey theorem.

✣ 8. Prove that there is an infinite set $S \subseteq \mathbb{N}$ such that for every $a, b \in S$, the number $a + b$ has an even number of prime factors (counted without multiplicity).