Exercises (recommended)

1. Prove that $r(K_{1,k}) = 2k$ if k is odd, and $r(K_{1,k}) = 2k - 1$ if k is even.

2. Let kK_2 denote a matching with k edges, that is, a disjoint union of k copies of the single-edge graph K_2. Prove that $r(kK_2) = 3k - 1$ for all $k \geqslant 1$.

3. (a) Prove that $r(T; q) \leqslant O(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.

   $\star$ (b) Prove that $r(T; q) = \Theta(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.

4. Prove that every non-empty forest has degeneracy 1.

5. Prove[1] that there exist absolute constants $C, c > 0$ such that the following holds for all n. There exists an n-vertex graph H with degeneracy $d \geqslant c \log_2 n$ and $r(H) \leqslant Cn$.

   Note that this result is close to optimal; by Theorem 15.8, such an upper bound on $r(H)$ cannot hold if $c > 2$.

Problems (optional)

1. Prove that for every integer k and for every n-vertex tree T, we have

$$r(K_k, T) = (k-1)(n-1) + 1.$$

$\star\star$ 2. Let P_k denote a k-vertex path. Prove that for all $k \geqslant \ell \geqslant 2$,

$$r(P_k, P_\ell) = k + \left\lfloor \frac{\ell}{2} \right\rfloor - 1.$$

3. (a) Prove that

$$r(C_{2k+1}; q) > 2^q k$$

   for all $k \geqslant 1, q \geqslant 2$.

   $\star$ (b) Prove that

$$r(C_{2k+1}; q) \leqslant C(q+2)!k,$$

   for some absolute constant C.

   ? (c) The previous two parts show that $r(C_{2k+1}; q)$ grows linearly in k and between exponentially and super-exponentially in q. Determine whether the true behavior is exponential or super-exponential.

$\star$ means that a problem is hard.

? means that a problem is open.

✣ means that a problem is on a topic beyond the scope of the course.

[1] *Hint:* Use a lot of isolated vertices.

$\star\star$ 4. Prove that $r(K_{k,k}) \leqslant O(2^k \log k)$.

5. For a bipartite graph H and a number $\delta > 0$, let $r_d(H; \delta)$ denote the minimum integer N such that every N-vertex graph with at least $\delta\binom{N}{2}$ edges has a copy of H.

   (a) Using what you know about extremal numbers of bipartite graphs, prove that $r_d(H; \delta)$ is well-defined, i.e. that this number is finite for all bipartite H and all $\delta > 0$.

   (b) By more carefully examining your solution to the previous part, show that for every bipartite graph H, there exists some $C > 0$ such that

$$r_d(H; \delta) \leqslant \left(\frac{1}{\delta}\right)^C$$

   for all $0 < \delta \leqslant \frac{1}{2}$.

   (c) Let H be a graph, and suppose G is an N-vertex graph with $\delta\binom{N}{2}$ edges and with no copy of H. Prove[2] that if q is an integer satisfying $(1-\delta)^q\binom{N}{2} < 1$, then

$$r(H; q) > N.$$

   (d) Fix a bipartite graph H, and let C be the constant from part (b). Using the previous parts, prove that

$$r_d\left(H; \frac{2C \ln q}{q}\right) \leqslant r(H; q) \leqslant r_d\left(H; \frac{1}{q}\right),$$

   This shows that $r(H; q)$ and $r_d(H; 1/q)$ are closely related for bipartite H. In particular, we see that Ramsey numbers of bipartite graphs are essentially controlled by extremal graph theory.

[2] *Hint:* Randomly permute the vertices of G to obtain q copies $G_1, \ldots, G_q$. Show that with positive probability, every edge of K_N appears in at least one G_i.