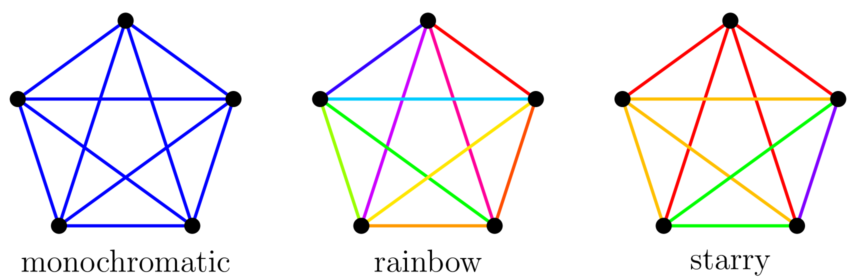
16.2 The canonical Ramsey theorem

We now turn to the canonical Ramsey theorem for edge-colorings of the complete graph. Of course, as discussed above, Ramsey's theorem itself is such a statement—any coloring of a complete graph with a fixed number of colors must contain an arbitrarily large monochromatic clique, and monochromatic cliques are clearly canonical, as any subset of a monochromatic clique is another monochromatic clique. However, what if we remove the restriction that the number of colors is fixed?

That is, the question we are asking is the following: we color $E(K_N)$, for a large N, with an arbitrary number of colors. What kinds of subcolorings are canonical, in the sense that all of their induced subgraphs yield colorings of the same type? Certainly, monochromatic cliques are still canonical. On the other hand, once the number of colors is unbounded, we get a new type of canonical coloring: a *rainbow* coloring of K_N, in which each of the edges receives a different color (so $\binom{N}{2}$ colors are used in total).

It is tempting to conjecture that these are the only ones, but this turns out to not be the case. There is a third type of coloring, which we will call *starry.* A coloring of $E(K_N)$ is called starry if there are distinct colors $c_1, \ldots, c_{N-1}$ and if one can sort the vertices as $v_1, \ldots, v_N$, such that the color of the edge v_iv_j, where $i < j$, is c_i. In other words, each color class is a star, with the first star centered at v_1, the second centered at v_2 (and not containing v_1), and so on. Note that this is a canonical coloring, as any subset of vertices induces another starry coloring.



As it turns out, these really are the only canonical colorings, in the sense that a canonical Ramsey theorem holds: every sufficiently large edge-colored clique contains an arbitrarily large clique which is monochromatic, rainbow, or starry. This was proved by Erdős and Rado, in a result that is now usually called *the* canonical Ramsey theorem.

Theorem 16.2 (Erdős–Rado)**.** *For every $k \geqslant 2$, there exists some N such that if $E(K_N)$ is colored (with an arbitrary number of colors), there is a K_k which is monochromatic, rainbow, or starry.*

The original proof of Erdős and Rado used a clever reduction to the hypergraph Ramsey theorem in uniformity 4. Namely, for every 4-tuple of vertices, they considered the equivalence relation of colors on the $\binom{4}{2} = 6$ edges. That is, rather than remembering the actual colors on each of these 6 edges, they only record which pairs of edges receive the same color. As it turns out, there are 203 equivalence relations[10] on a set of size 6, so they obtain a

[10] The number of equivalence relations on a set of size n is given by the *Bell number* B_n, and $B_6 = 203$.

203-coloring of $E(K_N^{(4)})$. By Theorem 14.1, there is a monochromatic $K_k^{(4)}$ in this coloring (assuming N is sufficiently large), and an elementary argument (involving some casework) shows that in each of the 203 cases[11], this monochromatic $K_k^{(4)}$ yields a monochromatic, rainbow, or starry K_k in the original coloring.

However, from a quantitative perspective, the proof of Erdős and Rado is not very good. Letting $\mathrm{ER}(k)$ denote the least N such that Theorem 16.2 holds, the proof of Erdős–Rado only shows that $\mathrm{ER}(k) \leqslant r_4(k; 203) \leqslant 2^{2^{2^{O(k)}}}$, thanks to the bounds on hypergraph Ramsey numbers. A much better bound, with an alternative proof that is also extremely elegant, was found by Lefmann and Rödl.

Theorem 16.3 (Lefmann–Rödl)**.** *We have* $\mathrm{ER}(k) \leqslant k^{4k^2}$ *for all* $k \geqslant 2$.

In particular, Theorem 16.3 gives a finite bound on $\mathrm{ER}(k)$, thus proving Theorem 16.2. In the course of the proof of Theorem 16.3, we will need the following extremely useful lemma, which allows us to find rainbow cliques in edge-colored graphs where every color class is a graph with bounded maximum degree.

Lemma 16.4. *Let* $k, M \geqslant 2$ *be integers, and suppose that* $E(K_M)$ *is colored so that every vertex is incident to at most* M/k^4 *edges in every color. Then there is a rainbow* K_k *in this coloring.*

Proof. Every vertex must be incident to at least one edge of some color, hence no such coloring can exist if $M < k^4$. Thus the statement is vacuously true in these cases, and we may assume henceforth that $M \geqslant k^4$. Also, since every coloring of $E(K_2)$ is rainbow, we may assume henceforth that $k \geqslant 3$. Let χ be the coloring of $E(K_M)$.

Let $v_1, \dots, v_k$ be a uniformly random sequence of k distinct vertices from K_M. That is, we pick a set of k distinct vertices uniformly at random among the $\binom{M}{k}$ options, and then pick a random ordering of that set and label it $v_1, \dots, v_k$. Equivalently, we let v_1 be a uniformly random vertex, v_2 a uniformly random vertex among the remaining vertices, and so on. The key property that we need about this distribution is that if we condition on the outcome of any subset of these vertices, the marginal distribution of any remaining vertex is that of a uniformly random vertex of K_M, apart the ones already picked. Thus, for example, if x, y are two distinct vertices of K_M, and we condition on $v_3 = x, v_7 = y$, the marginal distribution of v_4 is uniformly random on the set $V(K_M) \setminus \{x, y\}$.

For distinct indices $i, j, \ell \in [\![k]\!]$, let $\mathcal{E}_{i,j,\ell}$ be the event that the edges $v_i v_j$ and $v_i v_\ell$ receive the same color. We wish to estimate $\Pr(\mathcal{E}_{i,j,\ell})$. Given two distinct vertices $x, y \in V(K_M)$, we begin by estimating $\Pr(\mathcal{E}_{i,j,\ell} \mid v_i = x, v_j = y)$. Given $v_i = x, v_j = y$, the event $\mathcal{E}_{i,j,\ell}$ is simply the event that $\chi(xv_\ell) = \chi(xy)$, where the only randomness remaining is in the choice of v_ℓ. By assumption, x is incident to at most M/k^4 edges in color $\chi(xy)$, and v_ℓ is a uniformly random vertex in the set $V(K_M) \setminus \{x, y\}$, hence

$$\Pr(\mathcal{E}_{i,j,\ell} \mid v_i = x, v_j = y) \leqslant \frac{1}{M-2} \cdot \frac{M}{k^4} \leqslant \frac{2}{k^4}.$$

[11] In fact, it is not hard to show that most of the 203 cases are actually impossible, so the true number of cases is much smaller.

Since the same upper bound holds for $\Pr(\mathcal{E}_{i,j,\ell} \mid v_i = x, v_j = y)$ for all x, y, the same bound holds for $\Pr(\mathcal{E}_{i,j,\ell})$. More formally, by the law of total probability, we have

$$\Pr(\mathcal{E}_{i,j,\ell}) = \sum_{x,y} \Pr(\mathcal{E}_{i,j,\ell} \mid v_i = x, v_j = y) \Pr(v_i = x, v_j = y) \leqslant \frac{2}{k^4} \sum_{x,y} \Pr(v_i = x, v_j = y) = \frac{2}{k^4}.$$

Since the events $\mathcal{E}_{i,j,\ell}$ and $\mathcal{E}_{i,\ell,j}$ are the same, there are at most $k^3/2$ such events we need to consider. Hence, by the union bound, the probability that $\mathcal{E}_{i,j,\ell}$ occurs for some triple i, j, ℓ is at most $\frac{k^3}{2} \cdot \frac{2}{k^4} = \frac{1}{k} \leqslant \frac{1}{3}$.

Similarly, for four distinct indices i, j, ℓ, m, let $\mathcal{E}_{i,j,\ell,m}$ be the event that the edges v_iv_j and $v_\ell v_m$ receive the same color. For fixed vertices x, y, z, we now condition on the outcome $v_i = x, v_j = y, v_\ell = z$. By assumption, z has at most M/k^4 neighbors in color $\chi(xy)$. Once we condition, the event $\mathcal{E}_{i,j,\ell,m}$ is just the event that $\chi(zv_m) = \chi(xy)$, where the only randomness is in the choice of v_m, which is uniform on a set of size $M - 3$. So we have

$$\Pr(\mathcal{E}_{i,j,\ell,m} \mid v_i = x, v_j = y, v_\ell = z) \leqslant \frac{1}{M-3} \cdot \frac{M}{k^4} \leqslant \frac{2}{k^4}.$$

Again applying the law of total probability, we conclude that $\Pr(\mathcal{E}_{i,j,\ell,m}) \leqslant \frac{2}{k^4}$. The total number of such events is at most $k^4/4$, since we obtain the same event if we swap i, j or ℓ, m. So by the union bound, the probability that $\mathcal{E}_{i,j,\ell,m}$ happens for some 4-tuple (i, j, ℓ, m) is at most $\frac{k^4}{4} \cdot \frac{2}{k^4} = \frac{1}{2}$.

In total, we find that the probability that $v_1, \ldots, v_k$ span a rainbow K_k is at least $1 - \frac{1}{3} - \frac{1}{2} > 0$, hence there is a rainbow K_k in the coloring. □

Now that we have Lemma 16.4, we can proceed with the proof of Theorem 16.3. Before doing so, it's worth thinking of an alternative way of presenting the proof of Theorem 13.4. To show that $r(k) \leqslant 4^k$, let us fix a 2-coloring of $E(K_N)$, where $N = 4^k = 2^{2k}$. We pick an arbitrary vertex v_1. At least half of its incident edges are of the same color, which we call c_1. We now restrict to the c_1-colored neighborhood S_1 of v_1, and pick from there an arbitrary vertex v_2. At least half of its incident edges in S_1 are of the same color, say c_2. We let S_2 be this neighborhood, and proceed in this fashion. Since

$$|S_{i+1}| \geqslant \left\lceil \frac{|S_i| - 1}{2} \right\rceil$$

for all i, we conclude that $|S_i| \geqslant 2^{2k-i}$ for all i. Hence we can continue this process for at least $2k$ steps, to produce vertices $v_1, \ldots, v_{2k}$ and colors $c_1, \ldots, c_{2k}$. Again by the pigeonhole principle, at least k of these colors must be the same, say $c_{i_1}, \ldots, c_{i_k}$ are all red. But by the way we constructed this sequence, this shows that $v_{i_1}, \ldots, v_{i_k}$ form a red K_k.

The proof of Theorem 16.3 uses a very similar argument, which we will now see.

Proof of Theorem 16.3. Let $N = k^{4k^2}$, and fix an arbitrary coloring of $E(K_N)$. We let $S_0 = V(K_N)$. We now run the following process, for all $i \geqslant 1$.

1. If $|S_{i-1}| < 2$, stop the process.
2. If every vertex in S_{i-1} is incident to at most $|S_{i-1}|/k^4$ edges in each color, we apply Lemma 16.4 to S_{i-1} with $M = |S_{i-1}| \geqslant 2$. We conclude that S_{i-1} contains a rainbow K_k, completing the proof.
3. If not, there is some vertex $v_i \in S_{i-1}$ and some color c_i such that v_i is incident to at least $|S_{i-1}|/k^4$ edges of color c_i in S_{i-1}. We let S_i be the c_i-colored neighborhood of v_i in S_{i-1}.
4. Increment i by 1 and return to step 1.

If we ever find a rainbow K_k in this process, we are done, so we may assume that that never happens. Note that as long as the process continues, we have that $|S_i| \geqslant |S_{i-1}|/k^4$, so by induction we have that $|S_i| \geqslant k^{4(k^2-i)}$. Hence we can continue this process at least until step $i - 1 = k^2 - 1$. In other words, this process produces a sequence $v_1, \ldots, v_{k^2}$ of vertices and $c_1, \ldots, c_{k^2-1}$ of colors, with the property that each v_i is adjacent in color c_i to all v_j with $j > i$.

Suppose first that k of the colors $c_1, \ldots, c_{k^2-1}$ are equal, say $c_{i_1}, \ldots, c_{i_k}$ are all red. Then $v_{i_1}, \ldots, v_{i_k}$ form a monochromatic red K_k, and we are done. But if this does not happen, then at least k different colors must appear in the list $c_1, \ldots, c_{k^2-1}$, say $c_{j_1}, \ldots, c_{j_k}$ are all distinct. Then $v_{j_1}, \ldots, v_{j_k}$ form a starry K_k, and we are again done. □

Theorem 16.3 states that $\mathrm{ER}(k) \leqslant k^{4k^2} = 2^{4k^2 \log k}$. How good is this bound? The best known lower bound is given by the following simple proposition.

Proposition 16.5. *We have*

$$\mathrm{ER}(k) \geqslant r(k; k-2).$$

Proof. Let $N = r(k; k-2) - 1$, and consider a $(k-2)$-coloring χ of $E(K_N)$ with no monochromatic K_k. Note that a starry coloring of K_k must use $k-1$ colors, so there is also no starry K_k in χ, since χ only uses $k-2$ colors. Similarly, a rainbow coloring of K_k must use $\binom{k}{2} > k-2$ colors, hence there is no rainbow K_k in χ either. This shows that $\mathrm{ER}(k) > N$, proving the proposition. □

Note that this construction rules out the existence of starry or rainbow K_k in a pretty silly fashion, by simply using too few colors to allow these structures to appear. However, as far as I know, this is the only technique that anyone has ever found for lower-bounding $\mathrm{ER}(k)$; in particular, no one knows of a “smarter” way of excluding rainbow or starry K_k.

It now remains to find a good lower bound for the multicolor Ramsey number $r(k; k-2)$, or, more generally, for $r(k; q)$. By using a random q-coloring, one can adapt the proof of Theorem 13.7 and prove that for any $k, q \geqslant 3$, we have

$$r(k; q) > q^{k/2}. \tag{7}$$

However, this is not very good, as we recall that our upper bound on multicolor Ramsey numbers, from Theorem 13.5, is $r(k;q) \leqslant q^{qk}$; in particular, the dependence on q is super-exponential in the upper bound, whereas the lower bound in (7) is only polynomial in q. However, there is a simple construction that does substantially better.

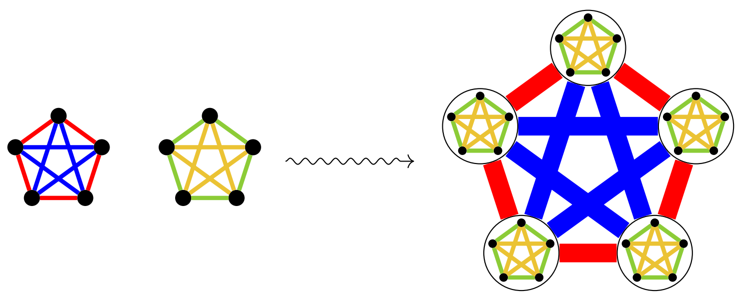
Proposition 16.6 (Abbott)**.** *For all positive integers k, q_1, q_2, we have*

$$r(k;q_1+q_2) - 1 \geqslant (r(k;q_1)-1)(r(k;q_2)-1). \tag{8}$$

As a consequence, we have

$$r(k;q) > 2^{\frac{k}{2}\lfloor \frac{q}{2} \rfloor}.$$

Proof. Let $N_1 = r(k;q_1) - 1$ and $N_2 = r(k;q_2) - 1$. By assumption, we have colorings $\chi_i : V(K_{N_i}) \to [\![q_i]\!]$, for $i = 1, 2$, both of which avoid monochromatic K_k. Let $N = N_1 N_2$, and identify the vertex set of K_N with $V(K_{N_1}) \times V(K_{N_2})$. We can now define a coloring $\chi : E(K_N) \to [\![q_1 + q_2]\!]$ as follows. It is easiest to understand with the following picture, which shows how to convert two 2-colorings of $E(K_5)$ into a 4-coloring of $E(K_{25})$, maintaining the property of having no monochromatic triangle.



Formally, given a pair of vertices $(a_1, b_1), (a_2, b_2) \in V(K_{N_1}) \times V(K_{N_2}) \cong V(K_N)$, we define

$$\chi((a_1,b_1),(a_2,b_2)) = \begin{cases} \chi_1(a_1,a_2) & \text{if } a_1 \neq a_2, \\ q_1 + \chi_2(b_1,b_2) & \text{otherwise.} \end{cases}$$

This is a (q_1+q_2)-coloring of $E(K_N)$, and one can readily verify that there is no monochromatic K_k, as such a monochromatic clique could be used to obtain a monochromatic K_k in either χ_1 or χ_2. Thus proves the claimed inequality (8).

To use it, we recall that we proved in Theorem 13.7 that $r(k;2) \geqslant 2^{k/2} + 1$. Applying (8) $\lfloor q/2 \rfloor$ times, we conclude that $r(k;q) > (2^{k/2})^{\lfloor q/2 \rfloor}$, as claimed. □

Plugging this result into Proposition 16.5, we find that $\mathrm{ER}(k) \geqslant r(k;k-2) \geqslant 2^{ck^2}$ for a constant $c > 0$. That is, we match the upper bound from Theorem 16.3, apart from a logarithmic gap in the exponent. In fact, we have the same logarithmic gap for multicolor Ramsey numbers, since we now know that $2^{ckq} \leqslant r(k;q) \leqslant 2^{kq \log q}$. It is a very major open problem to close this logarithmic gap in either problem.

Let me remark that in recent years, there have been a number of improvements to the constant factor in Proposition 16.6. Roughly speaking, it says that $r(k;q) \geqslant 2^{\frac{1}{4}kq}$. In

2020, Conlon and Ferber found a new construction that showed, roughly, $r(k;q) \geqslant 2^{\frac{7}{24}kq}$, which is better since $\frac{7}{24} > \frac{1}{4}$. Shortly thereafter, I optimized their technique and improved the lower bound to, roughly, $r(k;q) \geqslant 2^{\frac{3}{8}kq}$, which is again better since $\frac{3}{8} > \frac{7}{24}$. The current record is due to Sawin, who further optimized the technique and proved, roughly, that $r(k;q) \geqslant 2^{0.383796kq}$, which is better since $0.383796 > \frac{3}{8}$. Note that although these improvements are nice and interesting, they do not give any insight into the most important question of whether the logarithmic factor in the exponent is necessary, since they only affect the constant factor in the exponent.