17 Folkman's theorem and beyond

We started this topic with Ramsey's theorem: for every k, there exists an N such that if the edges of K_N are two-colored, then there exists a monochromatic K_k. In Section 15, we generalized the conclusion: rather than finding a monochromatic K_k, we found a monochromatic copy of H, for some not-necessarily-complete graph H. We will now generalize the first part of the statement.

Definition 17.1. Given two graphs G, H, we say that G *is Ramsey for* H *in* q *colors* (or G *is* q*-color Ramsey for* H) if, whenever the edges of G are q-colored, there is a monochromatic copy of H. In case $q = 2$, we simply say that G *is Ramsey for* H.

Thus, Ramsey's theorem simply states that K_N is q-color Ramsey for K_k whenever N is sufficiently large (as a function of q and k).

To gain some intuition for this definition, let's think of the case when $H = K_3$. If G is Ramsey for K_3, then certainly G must contain at least one triangle. But in fact, the definition of G being Ramsey for K_3 tells us that G contains triangles "very robustly". Indeed, another way of saying Definition 17.1 is that, no matter how we try to split G into the union of two subgraphs, we cannot destroy all triangles in G. This idea of robustness is one of the reasons that Definition 17.1 is interesting.

That being said, it's not at all obvious that this definition actually gives us any new information. Indeed, we know that $r(3) = 6$, or equivalently that K_6 is Ramsey for K_3 while K_5 is not. In particular, we find that if G is a graph containing K_6 as a subgraph, then G is Ramsey for K_3. Indeed, given any 2-coloring of $E(G)$, ignore all the edges except for those in the K_6 subgraph; among those $\binom{6}{2}$ edges, we are guaranteed to find a monochromatic triangle, regardless of how the other edges are colored.

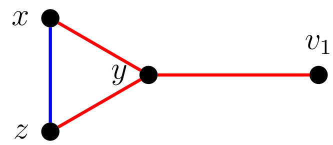
If you spend some time trying to construct graphs that are Ramsey for K_3, you may start to wonder if this is the *only* reason a graph can be Ramsey for K_3. In other words, you might be tempted to conjecture that G is Ramsey for K_3 if and only if $K_6 \subseteq G$. The question of whether this is true was raised by Erdős and Hajnal, and was rapidly answered in the negative independently by Cherlin, Graham, and van Lint. The following slick construction is due independently to Galluccio–Simonovits–Simonyi and to Szabó, and generalizes Graham's original argument. Given two graphs G_1, G_2, their *join*, denoted $G_1 * G_2$, is the graph obtained from their disjoint union by adding all edges with one endpoint in G_1 and one in G_2.

Proposition 17.2 (Galluccio–Simonovits–Simonyi, Szabó)**.** *Let* $G = K_3 * C_\ell$*, where* $\ell \geqslant 3$ *is an odd integer. Then* G *is Ramsey for* K_3*. Moreover, if* $\ell \geqslant 5$*, then* $K_6 \nsubseteq G$*.*

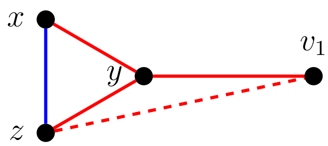
Proof. Let the vertices of G be $x, y, z, v_1, \ldots, v_\ell$, where x, y, z form a triangle, $v_1, \ldots, v_\ell$ form a cycle C_ℓ, and all edges between $\{x, y, z\}$ and $\{v_1, \ldots, v_\ell\}$ are present. Note that if $K_6 \subseteq G$, then at least three of the vertices of this K_6 must come from $v_1, \ldots, v_\ell$ (and they must form a triangle), so the second statement of the proposition is immediate since C_ℓ is triangle-free whenever $\ell \geqslant 5$.

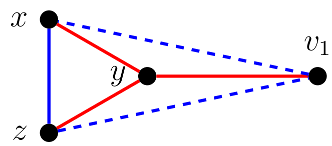
It remains to show that G is Ramsey for K_3, so fix a two-coloring of $E(G)$. If $\{x, y, z\}$ form a monochromatic triangle then we are done, so two of the edges xy, xz, yz receive one color and the third edge receives the other color. Without loss of generality, we may assume that xy, yz are red and xz is blue.

Now consider the edges between $\{x, y, z\}$ and v_1. First suppose yv_1 is red.

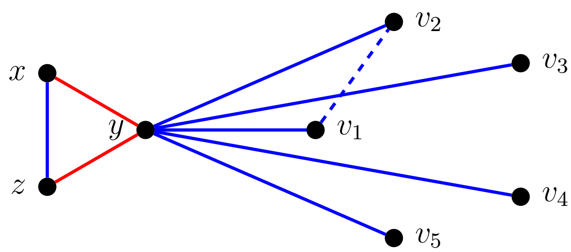


If xv_1 or zv_1 is red, then we close a red triangle xyv_1 or zyv_1, so we may assume that both these edges are blue. But that also creates a blue triangle, xzv_1.

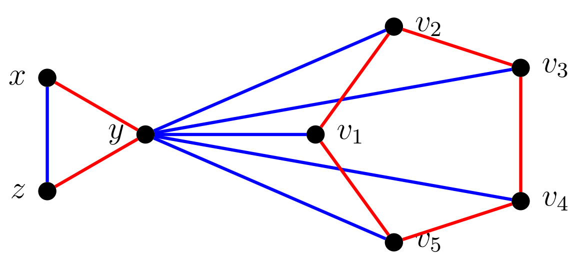




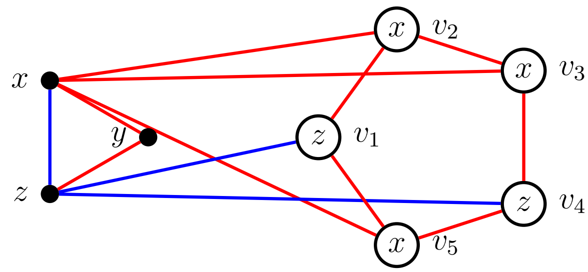
So we may assume that yv_1 is blue. By the same logic, yv_i is blue for all $i \in [\![\ell]\!]$. Note that if any of the edges v_iv_{i+1} in the cycle is blue, then we create a blue triangle yv_iv_{i+1}.



Therefore, we may assume that all the edges in the cycle are red.



Recall that xv_i and zv_i cannot both be blue, as this would create a blue triangle xzv_i. Let us label v_i by either the label x or z, depending on whether xv_i or zv_i is red (picking a label arbitrarily if both are red). By the above, every v_i receives a label.



Since ℓ is odd, the cycle C_ℓ is not bipartite. Hence, two adjacent vertices v_i, v_{i+1} must receive the same label (like v_2 and v_3 in the picture above). But then they create a red triangle together with their label. □

Note that $K_3 * K_3 = K_6$, so this result also gives a new (and more complicated) proof that K_6 is Ramsey for K_3. But it also shows that the set of graphs Ramsey for K_3 is surprisingly rich.

Note that each of the graphs $K_3 * C_\ell$ considered above *does* contain K_5 as a subgraph. So there is a natural weakening of our previous question: does every graph which is Ramsey for K_3 contain K_5 as a subgraph? The answer to this question also turns out to be negative, as proved by Pósa. So we may weaken our question further: does every graph Ramsey for K_3 contain a K_4? The answer to this also turns out to be no, as shown by the following remarkable theorem of Folkman.

Theorem 17.3 (Folkman)**.** *For every $k \geqslant 2$, there exists a graph G such that G is Ramsey for K_k, but $K_{k+1} \nsubseteq G$.*

This is pretty astonishing, even in the case $k = 3$. As discussed above, a graph that is Ramsey for K_3 must contain triangles "very robustly", in the sense that we cannot destroy all the triangles by splitting the graph into two subgraphs. Yet Folkman's theorem shows that such a graph can exist even though, locally, the triangles have almost no overlap.

Folkman's proof only worked for the case of two-colors, but the general case was shortly thereafter established by Nešetřil and Rödl, who proved the following generalization. We denote by $\omega(H)$ the *clique number* of H, that is, the maximum k such that $K_k \subseteq H$.

Theorem 17.4 (Nešetřil–Rödl)**.** *For every graph H and every $q \geqslant 2$, there exists a graph G which is q-color Ramsey for H with $\omega(G) = \omega(H)$.*

In their proof, Nešetřil and Rödl introduced a very powerful technique, called the *partite construction*, which is a very general-purpose way of producing graphs G that are Ramsey for a given graph H, while satisfying certain local sparsity conditions. The partite construction (as well as the earlier construction of Folkman) is completely explicit, so we can get a complete description of what the graph G in Theorem 17.4 looks like. Unfortunately, these constructions are iterative in nature, and each step of the iteration is complicated, so the *size* of the graph G constructed is unbelievably huge.

There is now an alternative approach to constructing such restricted Ramsey graphs, which uses randomness. It has a number of advantages over the partite construction, including giving much better bounds on how large G has to be in results like Theorem 17.4.

However, as we will discuss shortly, it also seems to be less flexible than the partite construction, and there are results that the random approach seems incapable of proving.

The main result in this direction is the *random Ramsey theorem* of Rödl and Ruciński. To state it, we recall that the *maximal* 2-*density* of a graph H is

$$m_2(H) \coloneqq \max_{\substack{J \subseteq H \\ v(J) \geqslant 3}} \frac{e(J)-1}{v(J)-2}.$$

Theorem 17.5 (Rödl–Ruciński)**.** *Let H be a graph which is not a forest, and let $q \geqslant 2$. There exist constants $C > c > 0$ such that the following holds. Form an N-vertex graph G by including each edge independently with probability p. Then*

$$\lim_{N \to \infty} \Pr(G \textit{ is Ramsey for } H \textit{ in } q \textit{ colors}) = \begin{cases} 1 & \textit{if } p \geqslant CN^{-1/m_2(H)}, \\ 0 & \textit{if } p \leqslant cN^{-1/m_2(H)}. \end{cases}$$

In other words, $p \asymp N^{-1/m_2(H)}$ is a *threshold* for the property of G being Ramsey for H. If p is substantially smaller than this value, then G is extremely unlikely to be Ramsey for H, whereas if p is substantially larger than this value, then G is extremely likely to be Ramsey for H. The heuristic reason why this value of p controls the threshold is the following. One can check that at this value, a typical edge of G lies in a constant number of copies of H[12]. Thus, if $p \leqslant cN^{-1/m_2(H)}$ for a small constant c, then the majority of edges of G lie in *zero* copies of H, and thus it is not surprising that G does not "robustly" contain H; we should be able to color $E(G)$ and destroy all copies of H. On the other hand, if $p \geqslant CN^{-1/m_2(H)}$ for a large constant C, then most edges of G lie in very many copies of H, and we expect a great deal of interaction between the copies, such that destroying all of them becomes impossible no matter how we color the edges. While this is a good heuristic explanation, turning it into a proof is substantially harder, and we will not do so in this course.

However, Theorem 17.5 does allow us to easily prove results along the lines of Theorem 17.3. One can actually prove Theorem 17.3 as a consequence of (a more precise version of) Theorem 17.5, but we will content ourselves with proving the following weakening of Theorem 17.3, which generalizes Proposition 17.2 (which corresponds to the case $k = 3, q = 2$).

Proposition 17.6. *For every $k \geqslant 3$ and $q \geqslant 2$, there exists a graph G which is q-color Ramsey for K_k, but $K_{k+3} \nsubseteq G$.*

Proof. We begin by observing that

$$\frac{e(K_k)-1}{v(K_k)-2} = \frac{\binom{k}{2}-1}{k-2} = \frac{k^2-k-2}{2(k-2)} = \frac{k+1}{2}.$$

It is not hard to check that $\frac{e(J)-1}{v(J)-2}$ is strictly smaller for any proper subgraph $J \subsetneq K_k$, hence $m_2(K_k) = \frac{k+1}{2}$. By Theorem 17.5, there is a constant $C > 0$ such that the following holds. If

[12] I am cheating a bit here; really, I should be counting copies of the subgraph $J \subseteq H$ achieving the maximum in the definition of $m_2(H)$.

we pick an N-vertex graph randomly by including each edge independently with probability $p \coloneqq CN^{-\frac{2}{k+1}}$, then G is q-color Ramsey for H with probability tending to 1 as $N \to \infty$. In particular, if N is sufficiently large, then this probability is at least $\frac{2}{3}$.

On the other hand, by the union bound, the probability that $K_{k+3} \subseteq G$ is at most

$$\binom{N}{k+3} p^{\binom{k+3}{2}} < C^{\binom{k+3}{2}} \cdot N^{k+3} \cdot N^{-\frac{2}{k+1}\binom{k+3}{2}} = C^{\binom{k+3}{2}} \cdot N^{-\left(\frac{2}{k+1}\binom{k+3}{2}-(k+3)\right)}. \tag{9}$$

We have that

$$\frac{2}{k+1}\binom{k+3}{2} - (k+3) = \frac{(k+3)(k+2)}{k+1} - (k+3) = (k+3)\left(\frac{k+2}{k+1} - 1\right) > 0.$$

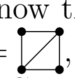
Hence, the exponent on N is negative in (9), so the probability that $K_{k+3} \subseteq G$ tends to 0 as $N \to \infty$. In particular, by picking N sufficiently large, we can ensure that $K_{k+3} \nsubseteq G$ with probability at least $\frac{2}{3}$.

Therefore, with positive probability, G satisfies both the desired properties, proving the claimed result. □

Before ending this section, let us briefly discuss one further recent breakthrough on the structure of restricted Ramsey graphs, due to Reiher and Rödl.

Definition 17.7. Let H be a graph. We say that another graph F is *Ramsey obligatory for* H if the following holds. For every sufficiently large q and every graph G which is q-color Ramsey for H, we have $F \subseteq G$.

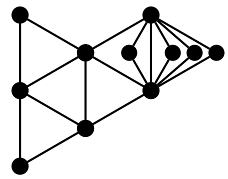
In this language, we can restate Proposition 17.6 as saying that K_{k+3} is not Ramsey obligatory for K_k, and Theorem 17.3 (or more precisely its multicolor extension, which follows from Theorem 17.4) states that K_{k+1} is not Ramsey obligatory for H. On the other hand, we can easily show that certain graphs *are* Ramsey obligatory for H. For example, H itself is Ramsey obligatory for H—if G is Ramsey for H, then certainly G contains H as a subgraph!

To keep things concrete, let's specialize to the case $H = K_3$. Then we know that K_3 is Ramsey obligatory for K_3, but K_4 is not. On the other hand, the graph $F =$ , obtained by gluing two triangles along an edge, is also Ramsey obligatory. Indeed, if G is an F-free graph, then all the triangles in G are edge-disjoint, so certainly we can color $E(G)$ and avoid all monochromatic triangles. More generally, we make the following definition.

Definition 17.8. *Triangle trees* are the class of graphs defined recursively as follows.

- K_3 is a triangle tree.
- Given a triangle tree T, we can obtain a new triangle tree T' by picking an edge of T and gluing a new triangle to it.

A typical triangle tree might look something like the following.



It is not hard to show the following fact; the proof is left for the homework.

Proposition 17.9. *If F is a subgraph of a triangle tree, then F is Ramsey obligatory for K_3.*

The astonishing theorem of Reiher and Rödl is that this sufficient condition is also necessary.

Theorem 17.10 (Reiher–Rödl)**.** *A graph F is Ramsey obligatory for K_3 if and only if F is a subgraph of a triangle tree.*

Said differently, given any graph F which is not a subgraph of a triangle tree, Reiher and Rödl are able to construct a graph G which is q-color Ramsey for K_3, yet does not contain F as a subgraph. In particular, since one can check that K_4 is not a subgraph of a triangle tree, this implies the $k = 3$ case of Theorem 17.3.

In fact, their theorem is vastly more general than this, and implies many strengthenings of Theorem 17.4. Somewhat more surprisingly, it appears that even for proving a result like Theorem 17.10, one actually has to prove these much more general results; their proof is based on a very complicated inductive argument, and in order to make the induction work one has to maintain a very general inductive statement.

18 Book recommendations

If you want to learn more about extremal combinatorics, Ramsey theory, or related topics, here are a few wonderful books.

- László Lovász, *Combinatorial problems and exercises*
- Yufei Zhao, *Graph theory and additive combinatorics*
- Dhruv Mubayi and Jacques Verstraete, *Extremal graph and hypergraph theory* (not yet published, but soon!)
- Ron Graham, Bruce Rothschild, and Joel Spencer, *Ramsey theory*
- Jiří Matoušek, *Thirty-three miniatures*
- Noga Alon and Joel Spencer, *The probabilistic method*
- My lecture notes on Ramsey theory (which were the origin of much of these lecture notes!): `https://n.ethz.ch/~ywigderson/math/static/RamseyTheory2024LectureNotes.pdf`