Before moving on, let me just mention one convenient way to think about Turán's theorem is as follows. Note that

$$\binom{n}{2} = \frac{n^2}{2} - \frac{n}{2} = (1+o(1))\frac{n^2}{2}.$$

This shows that

$$t_{r-1}(n) = \left(1 - \frac{1}{r-1} + o(1)\right)\frac{n^2}{2} = \left(1 - \frac{1}{r-1} + o(1)\right)\binom{n}{2}.$$

Note that an n-vertex graph can have anywhere between 0 and $\binom{n}{2}$ edges. So Turán's theorem implies that a K_r-free n-vertex graph can have at most, asymptotically, a $1-1/(r-1)$ fraction of all possible edges.

4 Beyond Turán's theorem

Turán's theorem is great, and tells us exactly what $\mathrm{ex}(n, K_r)$ is for any r. But we started this class by asking about $\mathrm{ex}(n, H)$ for general H; what can we say about that? In general, we'd probably expect this problem to be really hard, and the answer should depend in complicated ways on the fixed graph H.

But it turns out that's not the case! Kind of amazingly, the answer depends, essentially, on a single parameter of the graph H—its chromatic number.

Theorem 4.1 (Erdős–Stone(–Simonovits) 1946 (1966))**.** *For any graph H,*

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

Remark.

- This is sometimes called the Fundamental Theorem of Extremal Graph Theory, for hopefully obvious reasons: it more or less completely resolves the main question that we started with.
- The history (and naming) of this theorem is a bit confusing. Erdős and Stone proved a special case of it in 1946. In 1966, Erdős and Simonovits realized that the special case actually implies (with a one-line implication) the general case, which had not been really studied before. We will soon see the special case, and how it implies the general case.
- Notice that if H is bipartite (i.e. if $\chi(H) = 2$), then $1 - 1/(\chi(H) - 1) = 0$. So the theorem simply says that if H is bipartite, then

  $$\mathrm{ex}(n, H) = o(1) \cdot \binom{n}{2},$$

which we usually write as $\mathrm{ex}(n, H) = o(n^2)$. In other words, if G is an n-vertex graph containing no copy of some fixed bipartite graph H, then G must have *very few* edges—its number of edges grows sub-quadratically in n. Said differently, the fraction of all possible edges that we can put in such a graph is a vanishingly small fraction; the fraction tends to 0 as $n \to \infty$.

Already this statement is far from obvious, and we'll soon prove it. In fact, as we'll see, proving the statement for bipartite H implies, in a certain sense, the full Erdős–Stone–Simonovits theorem.

The next few lectures will be spent on proving the Erdős–Stone–Simonovits theorem. To do so, we'll prove upper and lower bounds on $\mathrm{ex}(n, H)$ of the form $(1-1/(\chi(H)-1)+o(1))\binom{n}{2}$. In fact, we can easily dispense with the lower bound.

Proposition 4.2. *For any fixed graph H and integer n,*

$$\mathrm{ex}(n, H) \geqslant t_{\chi(H)-1}(n) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}.$$

Proof. We claim that the Turán graph $T_{\chi(H)-1}(n)$ has no copy of H. Indeed, suppose we had some vertices in $T_{\chi(H)-1}(n)$ that defined a copy of H. Give the parts of $T_{\chi(H)-1}(n)$ names, say $V_1, \ldots, V_{\chi(H)-1}$. Then note that any two vertices of H that lie in the same part V_i cannot be adjacent in H, since $T_{\chi(H)-1}(n)$ has no edges inside any part V_i. Said differently, if we assign to any vertex v of H the number i so that $v \in V_i$, then two adjacent vertices are assigned different numbers. In other words, this yields a proper coloring of H with $\chi(H)-1$ colors. But this contradicts the definition of the chromatic number. □

5 Extremal numbers of bipartite graphs

5.1 Upper bounds

Let H be a bipartite graph. Recall that the Erdős–Stone–Simonovits theorem implies that in this case, $\mathrm{ex}(n, H) = o(n^2)$, or equivalently that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{n^2} = 0.$$

This is pretty surprising! For example, the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ has $\lfloor n^2/4\rfloor$ edges and no copy of any odd cycle $C_{2\ell+1}$. Thus, $\mathrm{ex}(n, C_{2\ell+1}) \geqslant \lfloor n^2/4\rfloor$ for all ℓ. But the four-cycle (or any other even cycle) is bipartite, so the Erdős–Stone–Simonovits theorem implies that $\mathrm{ex}(n, C_4) = o(n^2)$. What's up with that?

We will shortly prove that in fact, $\mathrm{ex}(n, C_4) \leqslant O(n^{3/2})$. In case you haven't seen it before, the big-O notation means that $\mathrm{ex}(n, C_4) \leqslant Cn^{3/2}$ for some absolute constant C, which we won't specify. In other words, we will shortly prove that if G is an n-vertex graph with at least $Cn^{3/2}$ edges, then G has a copy of C_4, assuming C is an appropriately chosen constant.

As a warm-up, we will begin with an easier special case of this result, which is the case when G is d-regular (i.e. every vertex in G has degree d). Recall that in any graph, the sum of the degrees of all the vertices equals twice the number of edges, so if G is d-regular then it has $dn/2$ edges. Thus, if G has $Cn^{3/2}$ edges and is d-regular, then $d = 2C\sqrt{n}$.

Proposition 5.1. *Let G be a d-regular n-vertex graph. If $d \geqslant 2\sqrt{n}$, then G contains a copy of C_4.*

Proof. Suppose for contradiction that G is C_4-free. We count the number of copies of $K_{1,2}$ in G, where $K_{1,2} =$ ●—●—● consists of one central vertex adjacent to two outer vertices. On the one hand, if we sum over all possibilities for the central vertex, we see that

$$\#(K_{1,2} \text{ in } G) = \sum_{v \in V(G)} \#(K_{1,2} \text{ with central vertex } v) = \sum_{v \in V(G)} \binom{\deg(v)}{2} = n\binom{d}{2}.$$

On the other hand, suppose we fix some $u, w \in V(G)$. We claim that they can be the outer vertices of at most one copy of $K_{1,2}$. Indeed, if not, then we would have two $K_{1,2}$s agreeing on the outer vertices, which yields a copy of C_4, a contradiction. So we conclude that

$$\#(K_{1,2} \text{ in } G) = \sum_{\substack{u,w \in V(G)\\ \text{distinct}}} \#(K_{1,2} \text{ with outer vertices } u, w) \leqslant \sum_{\substack{u,w \in V(G)\\ \text{distinct}}} 1 = \binom{n}{2}.$$

Rearranging, we see that

$$n\binom{d}{2} \leqslant \binom{n}{2} \qquad \iff \qquad \binom{d}{2} \leqslant \frac{n-1}{2} \qquad \iff \qquad d(d-1) \leqslant n-1.$$

But if $d \geqslant 2\sqrt{n}$ and $n \geqslant 0$, then this is a contradiction. □

To prove the real result, we will need one extraordinarily useful analytic tool, called *Jensen's inequality.* We will actually only need the following special case. For a real number x and a positive integer r, we extend the definition of the binomial coefficient as

$$\binom{x}{r} = \frac{x(x-1)(x-2)\cdots(x-r+1)}{r!}.$$

Lemma 5.2 ((Consequence of) Jensen's inequality)**.** *Let $r \geqslant 1$ be a positive integer, and let $x_1, \dots, x_n$ be non-negative integers. Suppose that $\frac{1}{n}\sum_{i=1}^n x_i \geqslant r$. Then*

$$\sum_{i=1}^n \binom{x_i}{r} \geqslant n\binom{\frac{1}{n}\sum_{i=1}^n x_i}{r}.$$

The point of this is that if we add up terms of the form $\binom{x_i}{r}$, we can only decrease the sum if we replace each x_i by the *average* of all the x_i. One says that the function $x \mapsto \binom{x}{r}$ is *convex*: the sum of its values is minimized when all the variables are equal (to their average).

We won't prove Jensen's inequality in class, but its proof is on the homework if you're interested. Once we have Jensen's inequality, we can easily prove the full result that $\mathrm{ex}(n, H) \leqslant O(n^{3/2})$. In fact, we will prove the following much more general result.

Theorem 5.3 (Kővári–Sós–Turán 1954). *For positive integers $s \leqslant t$, we have*

$$\mathrm{ex}(n, K_{s,t}) \leqslant O(n^{2-1/s}).$$

Here, the implicit constant may depend on s and t (which we think of as fixed).

Proof. We proceed much as in the proof of Proposition 5.1. Let G be an n-vertex graph with at least $Cn^{2-1/s}$ edges, where C is some large constant we will pick later. Let d be the average degree in G, so that $d = \frac{2}{n}e(G) \geqslant 2Cn^{1-1/s}$. Suppose for contradiction that G is $K_{s,t}$-free. We count the number of copies of $K_{1,s}$ in G in two ways. First, by summing over the options for the central vertex, we have that

$$\#(K_{1,s} \text{ in } G) = \sum_{v \in V(G)} \binom{\deg(v)}{s} \geqslant n\binom{d}{s},$$

using Lemma 5.2, as well as the fact that $d \geqslant s$ by picking C sufficiently large. On the other hand, by counting over the s outer vertices of $K_{1,s}$, we have that every $u_1, \dots, u_s \in V(G)$ can be the outer vertices of at most $t-1$ copies of $K_{1,s}$. So

$$\#(K_{1,s} \text{ in } G) \leqslant \sum_{\substack{u_1,\dots,u_s \in V(G) \\ \text{distinct}}} (t-1) = (t-1)\binom{n}{s}.$$

Combining these, we see that

$$(t-1)\binom{n}{s} \geqslant n\binom{d}{s} \qquad \iff \qquad (t-1)(n-1)(n-2)\cdots(n-s+1) \geqslant d(d-1)\cdots(d-s+1).$$

Now, if n is very large (which is the regime we care about anyway), all this subtracting stuff doesn't matter. So this is roughly equivalent to

$$(t-1)n^{s-1} \geqslant d^s \qquad \iff \qquad d \leqslant (t-1)^{1/s} n^{1-1/s}.$$

If C is sufficiently large, then this is a contradiction. Moreover, if C is sufficiently large, then the slightly sketchy step above where we dropped the subtractions is also OK, and we get the desired contradiction. □

Note that $C_4 = K_{2,2}$, so in the case $s = t = 2$, we indeed get the claimed bound of $\mathrm{ex}(n, C_4) \leqslant O(n^{3/2})$.