Despite not knowing the hypergraph analogues of Mantel's or Turán's theorems, the hypergraph analogue of the Kővári–Sós–Turán theorem *is* known, and is due to Erdős (1965). To state this, we need to define the hypergraph analogue of a bipartite graph.

Definition 6.4. A k-graph $\mathcal{H}$ is called *k-partite* if its vertex set can be split into k parts, so that every hyperedge of $\mathcal{H}$ contains exactly one vertex from each part.

The *complete k-partite k-graph with parts of sizes* $s_1, \ldots, s_k$, denoted $K^{(k)}_{s_1,\ldots,s_k}$, is the k-graph with parts of sizes $s_1, \ldots, s_k$, containing every edge with exactly one vertex from each part.

Note that in case $k = 2$, this simply recovers the definition of a bipartite graph and a complete bipartite graph. Because of this, the following result generalizes the Kővári–Sós–Turán theorem.

Theorem 6.5 (Erdős 1965)**.** *Let $s_1 \leqslant \cdots \leqslant s_k$ be positive integers. Then*

$$\mathrm{ex}(n, K^{(k)}_{s_1,\ldots,s_k}) \leqslant O\left(n^{k-\frac{1}{s_1 s_2 \cdots s_{k-1}}}\right).$$

The most important thing here is that any n-vertex k-graph has at most $\binom{n}{k} = \Theta(n^k)$ hyperedges. So this upper bound has a smaller exponent on n. This implies that if $\mathcal{H}$ is *any* k-partite k-graph, then

$$\mathrm{ex}(n, \mathcal{H}) = o(1) \cdot \binom{n}{k}.$$

The upper bound in Theorem 5.3 is still the best upper bound we have on extremal numbers of k-partite k-graphs. Moreover, as in the case of graphs, it is known that the bound in Theorem 6.5 is best possible if s_k is sufficiently large with respect to $s_1, \ldots, s_{k-1}$.

The proof of Theorem 6.5 very similar to that of Theorem 5.3, except that we combine it with an induction on k. To keep the notation from getting too crazy, we will only prove it in the case $k = 3$, which we will derive from the case $k = 2$, i.e. the Kővári–Sós–Turán theorem. Also, we will only prove it in the case $s_1 = s_2 = s_3 = s$, i.e. we will prove that

$$\mathrm{ex}(n, K^{(3)}_{s,s,s}) \leqslant O(n^{3-1/s^2}). \tag{2}$$

Hopefully you'll believe me (or convince yourself that it's true if you don't!) that the general result follows from the same technique, just with more bookkeeping.

Proof of (2). Let $\mathcal{G}$ be an n-vertex 3-graph with at least Cn^{3-1/s^2} hyperedges, for some constant $C > 0$ we will pick later. Suppose for contradiction that $\mathcal{G}$ is $K^{(3)}_{s,s,s}$-free. Let X be the number of copies of $K^{(3)}_{1,1,s}$ in $\mathcal{G}$. We bound X in two ways.

First, for a pair of distinct vertices v, w, let $\mathrm{codeg}(v, w)$ denote the number of hyperedges containing both v and w. Then we first claim that

$$\sum_{\substack{v,w \in V(\mathcal{G}) \\ \text{distinct}}} \mathrm{codeg}(v, w) = 3e(\mathcal{G}).$$

This is true for the same reason that the sum of the degrees in a graph equals twice the number of edges. Namely, every hyperedge of G appears exactly three times in the sum on the left-hand side.

Using this, we see that by Jensen's inequality,

$$X = \sum_{\substack{v,w \in V(\mathcal{G}) \\ \text{distinct}}} \binom{\operatorname{codeg}(v,w)}{s} \geqslant \binom{n}{2} \binom{\frac{1}{\binom{n}{2}} \sum_{v,w} \operatorname{codeg}(v,w)}{s} = \binom{n}{2} \binom{3e(\mathcal{G})/\binom{n}{2}}{s}.$$

Note too that since $e(\mathcal{G}) \geqslant Cn^{3-1/s^2}$, we have that $3e(\mathcal{G})/\binom{n}{2} \geqslant \Omega(n^{1-1/s^2})$. This implies that

$$X \geqslant \binom{n}{2} \binom{3e(\mathcal{G})/\binom{n}{2}}{s} \geqslant cn^2 \cdot (Cn^{1-1/s^2})^s = cC^s n^{2+s-1/s}$$

for some absolute constant $c > 0$, depending only on s.

On the other hand, we may upper-bound X by counting over s-sets of vertices which can be the "outer" vertices of the $K^{(3)}_{1,1,s}$. Namely, fix distinct $u_1, \ldots, u_s \in V(\mathcal{G})$. We define a new graph (note: not a hypergraph, a *graph*) $G(u_1, \ldots, u_s)$ as follows. The vertex set of $G(u_1, \ldots, u_s)$ is $V(\mathcal{G}) \setminus \{u_1, \ldots, u_s\}$. Moreover, given $v, w \in V(\mathcal{G}) \setminus \{u_1, \ldots, u_s\}$, we make vw an edge of $G(u_1, \ldots, u_s)$ if and only if $\{v, w, u_1, \ldots, u_s\}$ form a copy of $K^{(3)}_{1,1,s}$.

Now, for every choice of $u_1, \ldots, u_s$, we claim that $G(u_1, \ldots, u_s)$ is a $K_{s,s}$-free graph. Indeed, if we had a copy of $K_{s,s}$ in $G(u_1, \ldots, u_s)$, then we would find a copy of $K^{(3)}_{s,s,s}$ in $\mathcal{G}$, which is a contradiction. So by the Kővári–Sós–Turán theorem, we know that

$$e(G(u_1, \ldots, u_s)) \leqslant O(n^{2-1/s})$$

for every choice of distinct $u_1, \ldots, u_s \in V(\mathcal{G})$.

We can use this to upper-bound X, as follows. Note that $e(G(u_1, \ldots, u_s))$ is precisely the number of copies of $K^{(3)}_{1,1,s}$ that have $u_1, \ldots, u_s$ as the outer vertices. This implies that

$$X = \sum_{\substack{u_1, \ldots, u_s \in V(G) \\ \text{distinct}}} e(G(u_1, \ldots, u_s)) \leqslant \sum_{\substack{u_1, \ldots, u_s \in V(G) \\ \text{distinct}}} O(n^{2-1/s}) = \binom{n}{s} \cdot O(n^{2-1/s}) = O(n^{2+s-1/s}).$$

Combining our upper and lower bounds on X, we see that

$$cC^s n^{2+s-1/s} \leqslant O(n^{2+s-1/s}),$$

where both c and the implicit constant in the big-O depend only on s. Thus, if we pick C sufficiently large, this is a contradiction, and we conclude that $\mathcal{G}$ has a copy of $K^{(3)}_{s,s,s}$. □

7 Supersaturation

In this section, we discuss a special case of a very important phenomenon in extremal combinatorics, known as *supersaturation*. Roughly speaking, extremal combinatorics proves results of the type "if some discrete structure is sufficiently 'large', then it contains at least one copy of some other structure". The example we've seen of this is Turán's theorem: if a graph (discrete structure) has sufficiently many edges (is large) then it contains a K_k subgraph (a copy of some other structure). Supersaturation, in general, boosts this to a statement of the type "if the discrete structure is just a bit larger, then it contains *very many* copies of the other structure". Specifically, we'll prove the following supersaturation version of Turán's theorem. It was first explicitly stated by Erdős and Simonovits in 1983, but it can implicitly be found in earlier works, e.g. of Erdős from 1971.

Theorem 7.1. *For every integer $k \geqslant 3$ and every real number $\varepsilon > 0$, there exists some $\delta > 0$ so that the following holds for all sufficiently large n. If G is an n-vertex graph with*

$$e(G) \geqslant \left(1 - \frac{1}{k-1} + \varepsilon\right)\binom{n}{2},$$

then G contains at least $\delta\binom{n}{k}$ copies of K_k.

Note that G has *at most* $\binom{n}{k}$ copies of K_k, so this theorem is pretty remarkable: it says that once we have just barely more edges than the Turán graph, we have not only one copy of K_k, but a constant proportion of *all possible* copies of K_k. To prove this theorem, we need the following useful lemma, which is stated in greater generality than we need.

For a graph G and a subset $M \subseteq V(G)$, we denote by $e(M)$ the number of edges entirely contained in M, or equivalently the number of edges in the induced subgraph $G[M]$.

Lemma 7.2. *Let $0 < \alpha < \beta < 1$ be real numbers, let $m \geqslant 2$ be an integer, and let G be an n-vertex graph with $n \geqslant m$. Assume that $e(G) \geqslant \beta\binom{n}{2}$. Then the number of sets $M \subseteq V(G)$ with $|M| = m$ and $e(M) \geqslant \alpha\binom{m}{2}$ is at least $(\beta - \alpha)\binom{n}{m}$.*

Proof. The key identity which underlies this proof is

$$\binom{n-2}{m-2} e(G) = \sum_{\substack{M \subseteq V(G) \\ |M| = m}} e(M).$$

This has a simple bijective proof. On the right-hand side, every edge uv is counted a number of times, and that number of times is simply the number of m-sets M which contain both u and v. But the number of such m-sets is exactly $\binom{n-2}{m-2}$, yielding the formula.

Now, let $\mathcal{M}_0$ denote the set of M with $e(M) < \alpha\binom{m}{2}$, and let $\mathcal{M}_1$ denote the set of M with $e(M) \geqslant \alpha\binom{m}{2}$. So our goal is to prove a lower bound on $|\mathcal{M}_1|$. Continuing the identity above, we can write

$$\begin{aligned}
\binom{n-2}{m-2} e(G) &= \sum_{M \in \mathcal{M}_0} e(M) + \sum_{M \in \mathcal{M}_1} e(M) \\
&\leqslant \sum_{M \in \mathcal{M}_0} \alpha\binom{m}{2} + \sum_{M \in \mathcal{M}_1} \binom{m}{2} \\
&= \binom{m}{2}\left(\alpha|\mathcal{M}_0| + |\mathcal{M}_1|\right)
\end{aligned}$$

since every m-set in $\mathcal{M}_0$ has at most $\alpha\binom{m}{2}$ edges, and every m-set in $\mathcal{M}_1$ has at most $\binom{m}{2}$ edges.

Note that $|\mathcal{M}_0| + |\mathcal{M}_1| = \binom{n}{m}$. Let $x = |\mathcal{M}_1|/\binom{n}{m}$, so that $1 - x = |\mathcal{M}_0|/\binom{n}{m}$. Dividing by $\binom{n}{m}\binom{m}{2}$, the above inequality yields

$$\frac{\binom{n-2}{m-2}}{\binom{n}{m}\binom{m}{2}} e(G) \leqslant \alpha(1-x) + x = \alpha + (1-\alpha)x.$$

Now, we recall that $e(G) \geqslant \beta\binom{n}{2}$, so

$$\frac{\binom{n-2}{m-2}}{\binom{n}{m}\binom{m}{2}} e(G) \geqslant \frac{\binom{n-2}{m-2}\binom{n}{2}}{\binom{n}{m}\binom{m}{2}} \beta.$$

The final step is another magic identity, which is that $\binom{n-2}{m-2}\binom{n}{2} = \binom{n}{m}\binom{m}{2}$; in other words, the complicated fraction above is simply equal to 1. Indeed, both sides of this identity count the same object, which is the number of ways of picking an m-set out of n objects, and then picking 2 objects from the m-set.

Combining all these inequalities, we find that

$$\beta \leqslant \alpha + (1-\alpha)x \qquad \Longleftrightarrow \qquad x \geqslant \frac{\beta - \alpha}{1 - \alpha},$$

which implies that

$$|\mathcal{M}_1| = x\binom{n}{m} \geqslant \frac{\beta-\alpha}{1-\alpha}\binom{n}{m} \geqslant (\beta - \alpha)\binom{n}{m},$$

as claimed. □