15 Graph Ramsey numbers

15.1 Introduction

We will now move away to a more general topic than we have considered so far, that of *graph Ramsey numbers.*

Definition 15.1. Given graphs $H_1, \dots, H_q$, their *Ramsey number* $r(H_1, \dots, H_q)$ is defined as the minimum N such that any q-coloring of $E(K_N)$ contains a monochromatic copy of H_i in color i, for some $i \in [\![q]\!]$. Here, by a monochromatic copy, we mean a subgraph of K_N isomorphic to H_i, all of whose edges receive color i.

In case $H_1 = \cdots = H_q = H$, we denote this Ramsey number by $r(H; q)$. In case $q = 2$, we use the shorthand $r(H) \coloneqq r(H; 2)$.

Of course, everything we have studied so far is a special case of these more general graph Ramsey numbers, as $r(k)$ is simply $r(K_k)$, and $r(k, \ell) = r(K_k, K_\ell)$, etc. However, it turns out that there is an extremely rich theory of Ramsey numbers of graphs H which are not necessarily complete graphs; moreover, most of the interesting results actually arise when H is extremely far from being a complete graph.

We begin with a simple observation, which is that if H_i is a subgraph of H_i', then $r(H_1, \dots, H_q) \leqslant r(H_1', \dots, H_q')$, since any monochromatic copy of H_i' also yields a monochromatic copy of H_i. Thus, $r(H) \leqslant r(H')$ whenever $H \subseteq H'$. Since every n-vertex graph is a subgraph of K_n, we conlude that

$$r(H) \leqslant r(K_n) < 4^n \qquad \text{for every } n\text{-vertex graph } H.$$

Thus, in the worst case, an n-vertex graph may have Ramsey number that is exponential in n.

On the other hand, the most general lower bound we can get is that $r(H) \geqslant n$ if H is an n-vertex graph. Indeed, we need at least n vertices to be able to "fit" a copy of H. Moreover, this trivial lower bound is best possible in general, for if H has no edges (or even one edge), then $r(H) = n$.

Thus, for a general n-vertex graph H, we know $n \leqslant r(H) \leqslant 4^n$, and both behaviors—linear in n and exponential in n—are possible, for the empty graph and the complete graph, respectively. Based on our experience for cliques, we might expect that the exponential bound should be closer to the truth for most graphs. However, the striking result that we will see is that for many "natural" classes of graphs—and, in fact, for all *sparse* graphs—the lower bound is much closer to the truth.

15.2 Ramsey numbers of trees

Let us begin with the following simple result, which was probably first observed by Erdős and Graham; it says that the lower bound is close to tight for trees.

Theorem 15.2. *If T is an n-vertex tree, then $r(T) \leqslant 4n - 3$.*

To prove this, we will use a simple lemma from graph theory, which you already saw on the homework.

Lemma 15.3. *Let T be an n-vertex tree. If G is a graph with minimum degree at least $n-1$, then $T \subseteq G$.*

Proof. We proceed by induction on n, with the base case $n = 1$ being trivial since the only 1-vertex tree is a subgraph of every non-empty graph. Inductively, suppose this is true for all $(n-1)$-vertex trees. Let T' be obtained from T by deleting a leaf v, and let u be the unique neighbor of v in T. By the inductive hypothesis, $T' \subseteq G$, so let us pick a copy of T' in G, and let w be the vertex of G filling the role of u. As G has minimum degree at least $n-1$, w has at least $n-1$ neighbors, and at most $n-2$ of these neighbors were used in embedding the other $n-2$ vertices of T'. Thus, there is at least one unused neighbor of w, which means that we can extend the T'-copy to a T-copy by adding in this unused neighbor. □

With this lemma, it is straightforward to prove Theorem 15.2.

Proof of Theorem 15.2. Let $N = 4n-3$, and fix a 2-coloring of $E(K_N)$. Without loss of generality, we may assume that at least half the edges are red. Let $G \subseteq K_N$ be the graph comprising the red edges, so

$$e(G) \geqslant \frac{1}{2}\binom{N}{2} = \frac{1}{2} \cdot \frac{(4n-3)(4n-4)}{2} = N(n-1).$$

By Lemma 10.3, there is a subgraph $G' \subseteq G$ of minimum degree at least $n-1$. By Lemma 15.3, we have $T \subseteq G'$, and this yields a monochromatic red copy of T. □

15.3 Ramsey numbers of complete bipartite graphs

We now turn our attention to complete bipartite graphs $K_{s,t}$.

Theorem 15.4. *For any $s \leqslant t$, we have*

$$r(K_{s,t}) \leqslant 2^{s+1}t.$$

Note that, if we plug in $s = t = n$, then we obtain that $r(K_{n,n}) = O(n2^n)$. Since $K_{n,n}$ has $2n$ vertices, this is a much better, although still exponential, bound than the naïve one of

$$r(K_{n,n}) \leqslant r(2n) < 4^{2n} = 16^n.$$

We remark that $r(K_{n,n})$ really does grow exponentially in n, and that the lower bound

$$r(K_{n,n}) > 2^{\frac{n-1}{2}}$$

will follow from a more general result, Proposition 15.5, which we will prove shortly. On the other hand, if we think of s as a constant, we obtain that $r(K_{s,t}) = O_s(t)$ as $t \to \infty$. Since $K_{s,t}$ has $s + t \leqslant 2t$ vertices, this shows that for fixed s, $K_{s,t}$ has a Ramsey number which is linear in its number of vertices—the same behavior as we saw for trees.

The proof of Theorem 15.4 uses essentially the same strategy we used in proving the Kővári–Sós–Turán theorem.

Proof of Theorem 15.4. The case $s = 1$ follows from a homework problem; it also follows, up to an additive constant of 1, from Theorem 15.2, since $K_{1,t}$ is a tree. We henceforth assume that $t \geqslant s \geqslant 2$.

Let $N = 2^{s+1}t$, and fix a red/blue coloring of $E(K_N)$. For every vertex $v \in V(K_N)$, let $\deg_R(v), \deg_B(v)$ denote the red and blue degrees, respectively, of v. Let S denote the number of monochromatic copies of $K_{1,s}$ in the coloring. We can count S by summing over all N choices for the central vertex, and then picking s distinct neighbors; this shows that

$$S = \sum_{v \in V(K_N)} \left(\binom{\deg_R(v)}{s} + \binom{\deg_B(v)}{s} \right).$$

Note that $\deg_R(v) + \deg_B(v) = N - 1$ for every v, and that the sum $\binom{x}{s} + \binom{N-1-x}{s}$ is minimized[7] when $x = N - 1 - x$, i.e. $x = \frac{N-1}{2}$. Therefore, we find that

$$S \geqslant N \cdot 2\binom{\frac{N-1}{2}}{s}.$$

On the other hand, another way of counting S is by counting over all options for the s leaves of the star. Let us assume for contradiction that there is no monochromatic $K_{s,t}$. Then every s-set of vertices forms the set of leaves of fewer than t red stars $K_{1,s}$, and of fewer than t blue stars $K_{1,s}$. Thus,

$$S < 2t\binom{N}{s}.$$

Comparing our lower and upper bounds on S, we find that

$$2t\binom{N}{s} > 2N\binom{\frac{N-1}{2}}{s}$$

or equivalently

$$t \cdot N(N-1)\cdots(N-s+1) > N \cdot \frac{N-1}{2}\left(\frac{N-1}{2} - 1\right)\cdots\left(\frac{N-1}{2} - s + 1\right).$$

Rearranging, this is equivalent to

$$\frac{2^s t}{N} > \left(\frac{N-1}{N}\right)\left(\frac{N-3}{N-1}\right)\left(\frac{N-5}{N-2}\right)\cdots\left(\frac{N-2s+1}{N-s+1}\right) = \prod_{i=0}^{s-1} \frac{N-2i-1}{N-i}.$$

However, we have that

$$\prod_{i=0}^{s-1} \frac{N-2i-1}{N-i} = \prod_{i=0}^{s-1}\left(1 - \frac{i+1}{N-i}\right) \geqslant 1 - \sum_{i=0}^{s-1} \frac{i+1}{N-i} \geqslant 1 - \frac{2\binom{s+1}{2}}{N} \geqslant \frac{1}{2},$$

[7] This is again a special case of Jensen's inequality. This special case can also be proved directly without appealing to convexity.

where the second inequality uses that $N \geqslant 2s$, hence $N - i \geqslant N/2$ for all $i \leqslant s-1$, and the third inequality uses that $2\binom{s+1}{2} = (s+1)s \leqslant (s+1)t \leqslant 2^s t = N/2$, since $2^s \geqslant s+1$ for all $s \geqslant 2$. Putting this all together, we conclude that

$$\frac{2^s t}{N} > \frac{1}{2},$$

which contradicts our choice of N. This contradiction completes the proof. □

15.4 The Burr–Erdős conjecture

So far, we have seen several examples of graph Ramsey numbers, and observed different growth rates. First, we know that $r(K_n)$ grows exponentially in n. Similarly, $r(K_{n,n})$ grows exponentially in n (and thus in $2n$, which is its number of vertices). On the other hand, all trees, as well as complete bipartite graphs in which one side has constant size, have Ramsey numbers *linear* in the number of vertices. Can we figure out a general rule explaining these extremely different growth rates?

Looking at the examples above, it is natural to guess that the major difference has to do with *density*. Both K_n and $K_{n,n}$ are very dense graphs, namely graphs with a quadratic number of edges. On the other hand, trees and complete bipartite graphs with one side of constant size are very sparse, in that their number of edges is only linear in their number of vertices. Equivalently, the average degree of the former graphs is large—linear in the number of vertices—whereas it is *constant* for the latter graphs. Perhaps this explains the difference in the Ramsey numbers?

As it turns out, this is close to the correct explanation. One direction really is true; if a graph has high average degree, then its Ramsey number is large, as shown in the following simple proposition.

Proposition 15.5. *If H has average degree d, then $r(H) > 2^{\frac{d-1}{2}}$.*

Proof. The proof is very similar to that of Theorem 13.7. Let H have $k \geqslant 2$ vertices, and thus $kd/2$ edges. Let $N = 2^{\frac{d-1}{2}}$, and consider a uniformly random 2-coloring of $E(K_N)$. Every tuple of k vertices in K_N forms a monochromatic copy of H with probability $2^{1-kd/2}$, and there are $k!\binom{N}{k}$ such tuples[8]. Therefore, the probability that the coloring has a monochromatic copy of H is at most

$$k!\binom{N}{k} \cdot 2^{1-\frac{kd}{2}} < N^k \cdot 2^{1-\frac{kd}{2}} = 2^{k\frac{d-1}{2}+1-\frac{kd}{2}} = 2^{1-\frac{k}{2}} \leqslant 1,$$

and thus there must exist a coloring with no monochromatic copies of H. □

[8]Note that we include an extra factor of $k!$, which was not present in the proof of Theorem 13.7. The reason is that K_k is highly symmetric; for a general H, we need to consider not only the k vertices that can define it, but also the potentially $k!$ different ways of identifying $V(H)$ with these k vertices.

Thus, we find that if H has average degree which is linear in its number of vertices $v(H)$, then $r(H)$ is exponential in $v(H)$. Is it possible that the same holds at the opposite extreme, namely that if H has constant average degree, then $r(H)$ is linear in $v(H)$, as happened for trees and complete bipartite graphs? It is not hard to see that the answer is no.

Proposition 15.6. *There exists an n-vertex graph H with average degree at most 1 and with $r(H) > 2^{\sqrt{n}/2}$.*

Proof. Let H be obtained from a complete graph $K_{\sqrt{n}}$ by adding $n - \sqrt{n}$ isolated vertices. Then H has $\binom{\sqrt{n}}{2}$ edges, and thus average degree $\frac{2}{n}\binom{\sqrt{n}}{2} \leqslant 1$. However,

$$r(H) \geqslant r(K_{\sqrt{n}}) > 2^{\sqrt{n}/2},$$

by Theorem 13.7. □

Given this example, it's clear why the naïve conjecture "constant average degree implies linear Ramsey number" cannot be true. Namely, the graph H above has constant average degree, but it contains a subgraph (namely $K_{\sqrt{n}}$) with much higher average degree, and it is this subgraph that really determines $r(H)$. This shows that rather than considering the global average degree, we need to consider a more refined parameter that takes into account subgraphs that are denser than H itself. There are several different ways of formalizing such a parameter, and they end up giving essentially identical results; we will use the following.

Definition 15.7. The *degeneracy* of a graph H is defined as the maximum, over all subgraphs $H' \subseteq H$, of the minimum degree of H'. H is said to be *d-degenerate* if its degeneracy is at most d.

From Lemma 10.3, we see that a d-degenerate graph has average degree at most $2d$. On the other hand, the H in Proposition 15.6 is an example of a graph with constant average degree and degeneracy $\sqrt{n} - 1$. Thus, we see that having bounded degeneracy is a strictly stronger condition than having bounded average degree. In particular, Proposition 15.5 implies that graphs with high degeneracy have large Ramsey numbers, as shown in the following result.

Theorem 15.8. *Let H be a graph of degeneracy d. Then $r(H) > 2^{\frac{d-1}{2}}$.*

Proof. By the definition of degeneracy, there is a subgraph $H' \subseteq H$ with minimum degree at least d, and thus also average degree at least d. Then Proposition 15.5 implies that

$$r(H) \geqslant r(H') > 2^{\frac{d-1}{2}}.$$ □

Given this, we can now amend our naïve conjecture to the following fundamental conjecture of Burr and Erdős.

Conjecture 15.9 (Burr–Erdős)**.** *Graphs of bounded degeneracy have linear Ramsey numbers.*

More precisely, for every $d \geqslant 1$ there exists $C \geqslant 1$ such that the following holds. If an n-vertex graph H is d-degenerate, then $r(H) \leqslant Cn$.

The Burr–Erdős conjecture is now a theorem.

Theorem 15.10 (Lee)**.** *Conjecture 15.9 is true.*

Now that we know that the Burr–Erdős conjecture is true, we can start asking more refined questions. What if the degeneracy is not bounded, but instead grows as a function of n? The following conjecture predicts a fairly precise answer.

Conjecture 15.11 (Conlon–Fox–Sudakov)**.** *If H has n vertices and degeneracy d, then*

$$r(H) = 2^{\Theta(d+\log n)}.$$

If d is much larger than $\log n$, then this conjecture predicts that $r(H)$ is exponential in d, matching the lower bound from Theorem 15.8. On the other hand, if d is much smaller than $\log n$ (e.g. if d is bounded, as in the Burr–Erdős conjecture), then it predicts that $r(H)$ is polynomial in n. While this conjecture remains open, there are a number of partial results that come quite close to proving it, differing from the conjecture by only a polylogarithmic factor in the exponent.